Uniqueness for closed embedded non-smooth hypersurfaces with constant anisotropic mean curvature

Miyuki Koiso*

Institute of Mathematics for Industry,
Kyushu University

Abstract

We discuss a variational problem for piecewise-smooth hypersurfaces in the $(n+1)$-dimensional euclidean space. The energy functional is an anisotropic energy which is a natural generalization of the area for surfaces. However, equilibrium hypersurfaces of this energy are not smooth in general. Locally they are solutions of a second order quasilinear PDE, which is elliptic in a part, and hyperbolic in the rest. In this article, we study the uniqueness problem for closed embedded equilibrium hypersurfaces and give an application to the anisotropic mean curvature flow. This article plays a role of an announcement of a part of the forthcoming paper [6].

1 Introduction

An anisotropic surface energy was introduced by J. W. Gibbs (1839-1903) in order to model the shape of small crystals ([19],[20]), which is defined as follows. Let $\gamma : S^n \to \mathbb{R}_{>0}$ be a positive continuous function on the unit sphere $S^n = \{\nu \in \mathbb{R}^{n+1} \mid \|\nu\| = 1\}$ in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Let X be a closed piecewise-C^2 hypersurface in $\mathbb{R}^{n+1}$ (the definition of piecewise-C^2 hypersurface will be given in §2). X will be represented as a piecewise-C^2 mapping $X : M \to \mathbb{R}^{n+1}$ from an n-dimensional oriented connected compact C^∞ manifold M into $\mathbb{R}^{n+1}$, and the unit normal vector field ν along X is defined on M except a set $S(X)$ with measure zero. Then, we define the anisotropic energy of X as $\mathcal{F}_\gamma(X) = \int_{M\setminus S(X)} \gamma(\nu)\, dA$, where dA

*This work was partially supported by JSPS KAKENHI Grant Number JP18H04487. 2010 Mathematics Subject Classification:49Q10, 53C45, 53C42, 53C44. Key words and phrases: anisotropic mean curvature, anisotropic surface energy, Wulff shape, Cahn-Hoffman vector field, anisotropic mean curvature flow.

is the n-dimensional volume form of M induced by X. If $\gamma \equiv 1$, $\mathcal{F}_\gamma(X)$ is the usual n-dimensional volume of X.

It is known that, for any positive number $V > 0$, among all closed hypersurfaces in $\mathbb{R}^{n+1}$ enclosing the $(n+1)$-dimensional volume V, there exists a unique minimizer $W(V)$ of $\mathcal{F}_\gamma$ ([17]). $W_\gamma(V_0)$ for the special value $V_0 := (n+1)^{-1} \int_{S^n} \gamma(\nu)\ dS^n$ is called the Wulff shape for γ, and we denote it by W_γ (the standard definition of the Wulff shape will be given in §2). If $\gamma \equiv 1$, W_γ is the unit sphere S^n. All $W_\gamma(V)$ are homothetic to W_γ. W_γ is convex but is not smooth in general.

A piecewise-C^2 hypersurface $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_\gamma$ for variations that preserve the enclosed $(n+1)$-dimensional volume (we call such a variation a volume-preserving variation) if and only if the anisotropic mean curvature Λ of X is constant on M and X satisfies a certain condition about its unit normal ν at its singular points (the details are given in §3). We call such X a CAMC (constant anisotropic mean curvature) hypersurface (Definition 3.2). If $\gamma \equiv 1$, the anisotropic mean curvature coincides with the mean curvature.

It is sometimes convenient to consider the homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ of γ that is defined by $\overline{\gamma}(rX) = r\gamma(X), \forall X \in S^n, \forall r \geq 0$. We say that γ is convex if $\overline{\gamma}$ is a convex function.

As we mentioned above, the closed energy-minimizer is a homothety of W_γ. Hence it is natural to ask whether any closed CAMC hypersurface a homothety of W_γ or not. The answer to this question is not affirmative even in the case where $\gamma \equiv 1$ ([18], [7], [8]). However, it is expected that, if a closed CAMC hypersurface satisfies a "good" condition, it is a homothety of W_γ. This is the main subject of this article, and we study the following question.

Question 1.1. *Is any closed embedded CAMC hypersurface a homothety of the Wulff shape?*

Here we say $X : M \to \mathbb{R}^{n+1}$ is embedded if X is an injective mapping. It is known that, if γ is of C^∞ and strictly convex (that is equivalent to the equation "$\Lambda \equiv$ constant" is strictly elliptic), the answer to this question is affirmative, which was proved by [1] for $\gamma \equiv 1$, and by [5] for general γ.

On the other hand, if γ has less convexity, the Wulff shape and CAMC hypersurfaces can have "edges" and we have the following "non-uniqueness" result.

Theorem 1.1 ([6])**.** *There exists a C^∞ function $\gamma : S^n \to \mathbb{R}_{>0}$ which is not convex such that there exist closed embedded CAMC hypersurfaces in $\mathbb{R}^{n+1}$ for γ each of which is not (any homothety or translation of) the Wulff shape.*

Theorems 1.1 is proved by giving examples (§4). The same examples are applied to the anisotropic mean curvature flow. In order to give the presice statement, we recall the Cahn-Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$ for γ which gives a representation of W_γ. ξ_γ is defined as

$$\xi_\gamma(\nu) := D\gamma|_\nu + \gamma(\nu)\nu = \overline{D}\overline{\gamma}|_\nu, \tag{1}$$

here $D\gamma$ is the gradient of γ on S^n and $\overline{D}$ is the gradient in $\mathbb{R}^{n+1}$. W_γ is a subset of the image $\xi_\gamma(S^n)$, and $W_\gamma = \xi_\gamma(S^n)$ holds if and only if γ is convex (cf. [10]). Let $X_t : M \to \mathbb{R}^{n+1}$ be a one-parameter family of piecewise-C^2 hypersurfaces with anisotropic mean curvature Λ_t and unit normal ν_t. Assume that the Cahn-Hoffman field $\tilde{\xi}_t = \xi_\gamma \circ \nu_t : M \setminus S(X) \to \mathbb{R}^{n+1}$ (which is an anisotropic generalization of the unit normal vector field. see §3) along X_t is defined on M. If X_t satisfies

$$\partial X_t/\partial t = \Lambda_t \tilde{\xi}_t,$$

it is called an anisotropic mean curvature flow, which diminishes the anisotropic energy if $\Lambda_t \not\equiv 0$ ([6]). By a simple observation, one can show the following.

Theorem 1.2 ([6]). *Let c be a positive constant. Set*

$$X_t := \sqrt{2(c-t)}\,\xi_\gamma, \quad t \le c.$$

Then X_t is a self-similar shrinking solution, that is
(i) $\partial X_t/\partial t = \Lambda_t \tilde{\xi}_t$, and
(ii) X_t is homothetic to ξ_γ and it shrinks as t increases.

By using this result and by giving suitable examples, the following non-uniqueness result is proved (§5).

Theorem 1.3 ([6]). *There exists a C^∞ function $\gamma : S^n \to \mathbb{R}_{>0}$ which is not convex such that there exist closed embedded self-similar shrinking solutions in $\mathbb{R}^{n+1}$ for γ each of which is homeomorphic to S^n and is not (any homothety or translation of) the Wulff shape.*

In contrast with this result, the round sphere is the only closed embedded self-similar shrinking solution of the mean curvature flow in $\mathbb{R}^3$ with genus zero ([4]).

We have given non-uniqueness results for non-convex γ. On the other hand, if γ is convex, one can expect that the uniqueness holds even if γ is not strictly convex. Here we mention this subject briefly. F. Morgan[12] proved that, if $\gamma : S^1 \to \mathbb{R}_{>0}$ is continuous and convex, any closed equilibrium rectifiable curve for $\mathcal{F}_\gamma$ in $\mathbb{R}^2$ with area constraint is a covering of a homothety of the Wulff shape (see [13] for another proof). In higher dimensions, it is expected that the following conjecture can be proved by a similar method to that given in [5].

Conjecture 1.1. *Assume that $\gamma : S^2 \to \mathbb{R}_{>0}$ is of C^2 and convex, then any closed embedded CAMC hypersurface is a homothety of the Wulff shape W_γ.*

This article is organized as follows. In §2 we give the formulation of piecewise-C^r hypersurfaces and the definition of the anisotropic energy for them. We also recall the definition of the Wulff shape. In §3, we give the definitions of various anisotropic curvatures (Definition 3.1). The first variation formula of the anisotropic surface energy (Proposition 3.1) and the Euler-Lagrange equations for our variational problem (Proposition 3.2) are given. In §4 we give examples which prove Theorems 1.1 and 1.3. In §5, we give an outline of the proofs of Theorems 1.2 and 1.3. In §6, we give some comments about uniqueness for closed stable CAMC hypersurfaces.

We should remark that Theorem 1.1, the examples given in §4, and Theorem 6.1 in this article give generalizations of Theorems 1.1-1.3 announced in [9].

2 Preliminaries

2.1 Definitions of piecewise-C^r immersion and its anisotropic energy

First we recall the definition of a *piecewise-C^r weak immersion*, ($r \in \mathbb{N}$), defined in [10]. Let $M = \cup_{i=1}^k M_i$ be an n-dimensional oriented compact connected C^∞ manifold, where each M_i is an n-dimensional connected compact submanifold of M with piecewise-C^∞ boundary, and $M_i \cap M_j = \partial M_i \cap \partial M_j$, $(i, j \in \{1, \cdots, k\}$, $i \neq j)$. We call a map $X : M \to \mathbb{R}^{n+1}$ a piecewise-C^r weak immersion if X satisfies the following conditions (A1), (A2), and (A3) for $i = 1, \cdots, k$.

(A1) X is continuous, and each $X_i := X|_{M_i} : M_i \to \mathbb{R}^{n+1}$ is of C^r.

(A2) The restriction $X|_{M_i^o}$ of X to the interior M_i^o of M_i is a C^r-immersion.

(A3) The unit normal vector field $\nu_i : M_i^o \to S^n$ along $X_i|_{M_i^o}$ can be extended to a C^{r-1}-mapping $\nu_i : M_i \to S^n$. Here the orientation of ν_i is determined so that, if $(u^1, \cdots, u^n)$ is a local coordinate system in M_i, then $\{\nu_i, \partial/\partial u^1, \cdots, \partial/\partial u^n\}$ gives the canonical orientation in $\mathbb{R}^{n+1}$.

Now let us fix a nonnegative continuous function $\gamma : S^n \to \mathbb{R}_{\geq 0}$. The anisotropic energy of a piecewise-C^1 weak immersion $X : M \to \mathbb{R}^{n+1}$ is defined as follows. Denote by $S(X)$ the set of all singular points of X, here a singular point of X is a point in M at which X is not an immersion. Let $\nu : M \setminus S(X) \to S^n$ be the unit normal vector field along $X|_{M \setminus S(X)}$. The anisotropic energy $\mathcal{F}_\gamma(X)$ of X is defined as

$$\mathcal{F}_\gamma(X) := \int_M \gamma(\nu)\, dA := \sum_{i=1}^k \int_{M_i} \gamma(\nu_i)\, dA. \tag{2}$$

Note that, since the n-dimensional Hausdorff measure of $X(S(X))$ is zero ([10]), each improper integral $\int_{M_i} \gamma(\nu_i)\, dA$ in (2) converges.

2.2 Wulff shape and convexity of integrands

Definition 2.1. Assume that S is a closed hypersurface in $\mathbb{R}^{n+1}$ that is the boundary of a bounded connected open set Ω. Denote by $\overline{\Omega}$ the closure of Ω. S is said to be convex (resp. strictly convex) if, for any straight line segment PQ connecting two points P and Q in S, $PQ \subset \overline{\Omega}$ (resp. $PQ \subset \overline{\Omega}$ and $PQ \cap S = \{P, Q\}$) holds.

For a continuous function $\gamma : S^n \to \mathbb{R}_{>0}$, the boundary W_γ of the convex set $\tilde{W}[\gamma] := \cap_{\nu \in S^n} \{X \in \mathbb{R}^{n+1} | \langle X, \nu \rangle \leq \gamma(\nu)\}$ is called the Wulff shape for γ, where $\langle \ , \ \rangle$ stands for the standard inner product in $\mathbb{R}^{n+1}$. We should remark that originally $\tilde{W}[\gamma]$ itself was called the Wulff shape.

If the homogeneous extension $\overline{\gamma}$ of γ is convex (that is, $\overline{\gamma}(X+Y) \leq \overline{\gamma}(X) + \overline{\gamma}(Y)$, $X, Y \in \mathbb{R}^{n+1}$) and satisfies $\overline{\gamma}(-X) = \overline{\gamma}(X)$, then $\overline{\gamma}$ defines a norm on $\mathbb{R}^{n+1}$, and the unit sphere $\{Y \in \mathbb{R}^{n+1} \mid \overline{\gamma}^*(Y) = 1\}$ of the dual norm $\overline{\gamma}^*(Y) = \sup\{\langle Y, Z \rangle \mid \overline{\gamma}(Z) \leq 1\}$ coincides with W_γ.

W_γ is smooth and strictly convex if and only if γ is of C^2 and the $n \times n$ matrix $D^2\gamma + \gamma \cdot I_n$ is positive definite at any point in S^n, where $D^2\gamma$ is the Hessian of γ on S^n and I_n is the identity matrix of size n. Recall that γ is said to be convex if its homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ is a convex function. γ is convex if and only if $D^2\gamma + \gamma \cdot I_n$ is positive semi-definite.

3 First variation formula, anisotropic curvatures, and anisotropic Gauss map

From now on, we assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2. Let $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ be a piecewise-C^2 weak immersion. The Cahn-Hoffman field $\tilde{\xi}_i$ along $X_i = X|_{M_i}$ for γ (or the anisotropic Gauss map of X for γ) is defined as $\tilde{\xi}_i := \xi_\gamma \circ \nu_i : M_i \to \mathbb{R}^{n+1}$. The linear map $S_p^\gamma : T_pM_i \to T_pM_i$ given by the $n \times n$ matrix $S^\gamma := -d\tilde{\xi}_i$ is called the anisotropic shape operatior of X_i.

Definition 3.1 (anisotropic principal curvatures and anisotropic mean curvature, cf. [16], [5])**.** (i) The eigenvalues of S^γ are called the anisotropic principal curvatures of X. We denote them by $k_1^\gamma, \cdots, k_n^\gamma$.

(ii) $\Lambda := (1/n)(k_1^\gamma + \cdots + k_n^\gamma)$ is called the anisotropic mean curvature of X.

Remark 3.1. At any regular point of X, it holds that (cf. [11])

$$\Lambda = -\frac{1}{n}\mathrm{trace}_M(D^2\gamma + \gamma \cdot 1) \circ d\nu = -\frac{1}{n}\mathrm{trace}_M\, d(\tilde{\xi}_\gamma). \tag{3}$$

Remark 3.2. Let X be a graph $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$ of a C^∞ function $f : D(\subset \mathbb{R}^2) \to \mathbb{R}$. Then, by using the homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ of γ, we have

$$\Lambda = (1/2) \sum_{i,j=1,2} \overline{\gamma}_{x_i x_j}\Big|_{(-Df,1)} f_{u_i u_j}, \quad Df := (f_1, f_2).$$

Hence, the equation "$\Lambda \equiv$ constant" is elliptic or hyperbolic depends on $(\overline{\gamma}_{x_i x_j}\Big|_{(-Df,1)})_{i,j=1,2}$.

Proposition 3.1 ([10])**.** *Assume that the map $X : M_0 \to \mathbb{R}^{n+1}$ satisfies (A1), (A2), and (A3) in §2 with $r = 2$, $X_i = X$, $M_i = M_0$, and $\nu_i = \nu$. Let $X_\epsilon : M_0 \to \mathbb{R}^{n+1}$,*

($\epsilon \in J := [-\epsilon_0, \epsilon_0]$), be a variation of X, that is, $\epsilon_0 > 0$ and $X_0 = X$. Assume for simplicity that X_ϵ is of C^∞ in ϵ. We also assume that, for each $\epsilon \in J$, the anisotropic mean curvature Λ_ϵ of X_ϵ is bounded on M_0^o. Set

$$\delta X := \frac{\partial X_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \psi := \langle \delta X, \nu \rangle.$$

Then the first variation of the anisotropic energy $\mathcal{F}_\gamma$ is given as follows.

$$\frac{d\mathcal{F}_\gamma(X_\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = -\int_{M_0} n\Lambda\psi \, dA - \oint_{\partial M_0} \langle \delta X, R(p(\tilde{\xi})) \rangle \, d\tilde{s}, \tag{4}$$

here N is the outward-pointing unit conormal along ∂M_0, $d\tilde{s}$ is the $(n-1)$-dimensional volume form of ∂M_0 induced by X, R is the $\pi/2$-rotation on the (N,ν)-plane, and p is the projection from $\mathbb{R}^{n+1}$ to the (N,ν)-plane.

On the other hand the first variation of the $(n+1)$-dimensional volume enclosed by X_ϵ is $\delta V = \int_{M_0} \psi \, dA$ (cf. [3]). This with (4) gives the following Euler-Lagrange equations.

Proposition 3.2 (Euler-Lagrange equations, Koiso [10]. For $n = 2$, see B. Palmer [15])**.** *A piecewise-C^2 weak immersion $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ is a critical point of the anisotropic energy $\mathcal{F}_\gamma$ for volume-preserving variations if and only if the following (i) and (ii) hold.*

(i) The anisotropic mean curvature of X is constant on $M \setminus S(X)$.

(ii) $\tilde{\xi}_i(\zeta) - \tilde{\xi}_j(\zeta) \in T_\zeta M_i \cap T_\zeta M_j = T_\zeta(\partial M_i \cap \partial M_j)$ holds at any $\zeta \in \partial M_i \cap \partial M_j$, where a tangent space of a submanifold of $\mathbb{R}^{n+1}$ is naturally identified with a linear subspace of $\mathbb{R}^{n+1}$.

Definition 3.2 ([10])**.** A piecewise-C^2 weak immersion $X : M = \cup_{i=1}^k M_i \to \mathbb{R}^{n+1}$ is called a hypersurface with constant anisotropic mean curvature (CAMC) if both of (i) and (ii) in Proposition 3.2 hold.

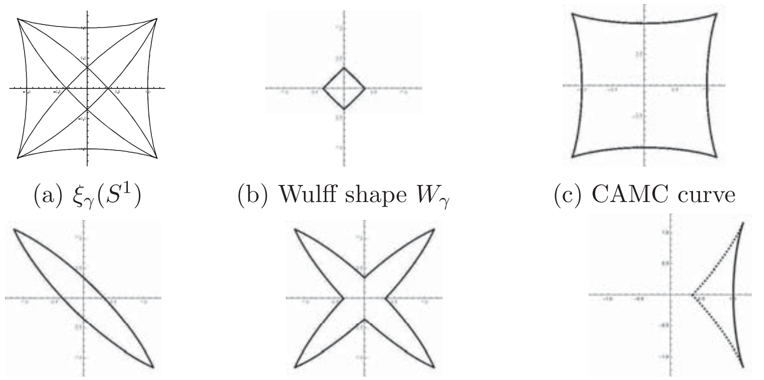
Fact 3.1 ([11], [10])**.** Since ξ_γ^{-1} gives the unit normal vector field ν_{ξ_γ} for the Cahn-Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$, the anisotropic shape operator of ξ_γ is $S^\gamma = -d(\xi_\gamma \circ \nu_{\xi_\gamma}) = -d(\mathrm{id}_{S^n}) = -I_n$. Hence, the anisotropic principal curvatures of ξ_γ are -1, and so the anisotropic mean curvature of ξ_γ with respect to ν and that of W_γ for the outward-pointing unit normal is -1 at any regular point.

4 Examples and proof of Theorem 1.1

The examples given in this section prove Theorem 1.1. Detailed explanation is given in [6].

Define $\gamma : S^1 \to \mathbb{R}_{>0}$ as

$$\gamma(\cos\theta, \sin\theta) := \cos^6\theta + \sin^6\theta, \quad (\cos\theta, \sin\theta) \in S^1. \tag{5}$$



(a) $\xi_\gamma(S^1)$ (b) Wulff shape W_γ (c) CAMC curve

(d) CAMC curve (e) CAMC curve (f) not CAMC

Figure 1: (a): Image of the Cahn-Hoffman map $\xi_\gamma(S^1)$ for γ defined by (5). (b) - (e) : The anisotropic curvature for the outward-pointing normal is -1. (f) : For the outward-pointing normal, the anisotropic curvature is -1 on the solid arc, while it is 1 on the dashed arcs.

Then, by using (1), we derive the following representation of the Cahn-Hoffman map $\xi_\gamma : S^1 \to \mathbb{R}^2$ for γ.

$$\begin{aligned} \xi_\gamma(\cos\theta, \sin\theta) &= \big((\cos\theta)(\cos^6\theta + 6\cos^4\theta\sin^2\theta - 5\sin^6\theta), \\ &\quad (\sin\theta)(-5\cos^6\theta + 6\cos^4\theta\sin^2\theta + \sin^6\theta)\big). \end{aligned} \tag{6}$$

The image $\xi_\gamma(S^1)$ is shown in Figure 1a. The Wulff shape W_γ shown in Figure 1b is its convex subset including the origin $(0,0)$ in the domain bounded by itself. Also Figures 1c - 1f are closed curves which are subsets of $\xi_\gamma(S^1)$. Because of Fact 3.1, on the closed curves shown in Figures 1c - 1e, the anisotropic (mean) curvature with respect to the outward-pointing unit normal is -1, and hence they are CAMC. On the other hand, the closed curve shown in Figure 1f is not CAMC, because, for the outward-pointing unit normal, the anisotropic curvature is -1 on the solid arc, while it is 1 on the dashed arcs.

We can construct higher dimensional examples by suitable rotations, for example, rotation around the vertical axis. Here we give only the case where $n = 2$.

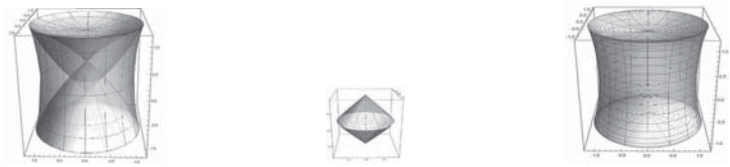
Define $\gamma_1 : S^2 \to \mathbb{R}_{>0}$ as

$$\gamma_1(\nu_1, \nu_2, \nu_3) = (\nu_1^2 + \nu_2^2)^3 + \nu_3^6, \quad (\nu_1, \nu_2, \nu_3) \in S^2. \tag{7}$$

The corresponding Cahn-Hoffman map $\xi_{\gamma_1} : S^2 \to \mathbb{R}^3$ is given as follows (Figure 2a).

$$\begin{aligned} \xi_{\gamma_1}(\nu) &= \big((\cos\theta)(\cos^6\theta + 6\cos^4\theta\sin^2\theta - 5\sin^6\theta)(\cos\rho), \\ &\quad (\cos\theta)(\cos^6\theta + 6\cos^4\theta\sin^2\theta - 5\sin^6\theta)(\sin\rho), \\ &\quad (\sin\theta)(-5\cos^6\theta + 6\cos^4\theta\sin^2\theta + \sin^6\theta)\big), \end{aligned} \tag{8}$$

($\nu = (\cos\theta\cos\rho, \cos\theta\sin\rho, \sin\theta) \in S^2$). The Wulff shape W_{γ_1} is the surface of revolution (Figure 2b) given by rotating W_γ (Figure 1b) around the vertical axis. The piecewise-C^∞ closed surfaces shown in Figures 2b - 2e are subsets of $\xi_{\gamma_1}(S^2)$. They are surfaces given by rotating the curves shown in Figures 1b, 1c, 1e, and 1f, respectively. From Fact 3.1, we have the following observation. The anisotropic mean curvature of the surfaces in Figures 2b - 2d for the outward-pointing normal is -1. On the surface in Figure 2e, it is -1 on the 'outer part', while it is 1 on the 'inner part'. Hence, this surface is not CAMC.



(a) $\xi_{\gamma_1}(S^2)$ (b) Wulff shape W_{γ_1} (c) CAMC surface

(d) CAMC surface (e) not CAMC

Figure 2: (a): The image of the Cahn-Hoffman map $\xi_{\gamma_1}(S^2)$ for γ_1 defined by (7). (b): Wulff shape W_{γ_1}. (c) and (d): Piecewise-C^∞ closed CAMC surfaces. (e): A piecewise-C^∞ closed non-CAMC surface.

Let us give another example. We rotate γ around the origin by $\pi/4$, and then rotate it around the vertical axis. Then, we obtain a new anisotropic energy density function $\gamma_2 : S^2 \to \mathbb{R}_{>0}$ which can be written as

$$\gamma_2(\nu_1,\nu_2,\nu_3) = (\nu_1^2+\nu_2^2)^3+15(\nu_1^2+\nu_2^2)^2\nu_3^2+15(\nu_1^2+\nu_2^2)\nu_3^4+\nu_3^6, \quad (\nu_1,\nu_2,\nu_3) \in S^2. \tag{9}$$

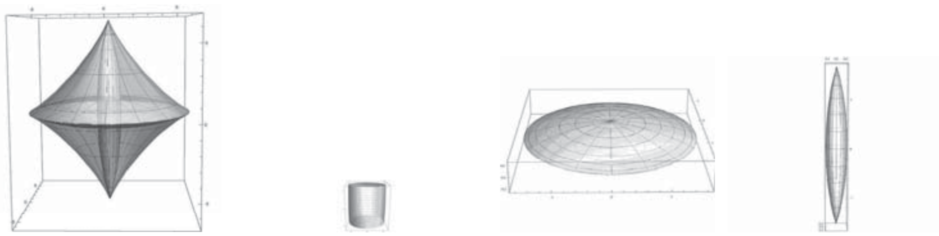
The corresponding Cahn-Hoffman map $\xi_{\gamma_2} : S^2 \to \mathbb{R}^3$ is given as follows (Figure 3a).

$$\begin{aligned}\xi_{\gamma_2}(\nu) = \frac{1}{4}\big(&(\cos\theta)(\cos^6\theta - 9\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta + 25\sin^6\theta)(\cos\rho),\\ &(\cos\theta)(\cos^6\theta - 9\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta + 25\sin^6\theta)(\sin\rho),\\ &(\sin\theta)(25\cos^6\theta + 15\cos^4\theta\sin^2\theta - 9\cos^2\theta\sin^4\theta + \sin^6\theta)\big),\end{aligned} \tag{10}$$

($\nu = (\cos\theta\cos\rho, \cos\theta\sin\rho, \sin\theta) \in S^2$).

By the same way as above, we obtain closed piecewise-C^∞ CAMC surfaces (Figures 3c, 3d) for γ_2 which are subsets of $\xi_{\gamma_2}(S^2)$ and which are not any homotheties of the

Wulff shape W_{γ_2} (Figure 3b). The anisotropic mean curvature of the surfaces in Figures 3b - 3d for the outward-pointing normal is -1.



(a) $\xi_{\gamma_2}(S^2)$ (b) Wulff shape W_{γ_2} (c) CAMC surface (d) CAMC surface

Figure 3: (a): The image of the Cahn-Hoffman map $\xi_{\gamma_2} : S^2 \to \mathbb{R}^3$ for $\gamma_2 : S^2 \to \mathbb{R}_{>0}$ defined by (9). (b): The Wulff shape W_{γ_2}. (c) and (d): Piecewise-C^∞ closed convex CAMC surfaces.

5 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Since the anisotropic mean curvature of ξ_γ is -1 (Fact 3.1), $\Lambda_t = \dfrac{-1}{\sqrt{2(c-t)}}$ holds. On the other hand, $\tilde{\xi}_t = \xi_\gamma$ holds. These two facts prove the desired results. □

Proof of Theorem 1.3. Examples given in §4 prove the desired result. □

6 Uniqueness for closed stable CAMC hypersurfaces

A CAMC hypersurface is said to be stable if the second variation of the energy $\mathcal{F}_\gamma$ for any volume-preserving variation with compact support is nonnegative. For convex γ, we have the following uniqueness result.

Theorem 6.1 ([10]). *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2 and convex. Then, the image of any closed stable piecewise-C^2 CAMC hypersurface for γ whose r-th anisotropic mean curvature for γ is integrable for $r = 1, \cdots, n$ is a coverng of a homothety of the Wulff shape W_γ.*

Here the r-th anisotropic mean curvatures are defined as follows. Let σ_r^γ be the elementary symmetric functions of the anisotropic principal curvatures $k_1^\gamma, \cdots, k_n^\gamma$:

$$\sigma_r^\gamma := \sum_{1 \le l_1 < \cdots < l_r \le n} k_{l_1}^\gamma \cdots k_{l_r}^\gamma, \quad r = 1, \cdots, n. \tag{11}$$

Set $\sigma_0^\gamma := 1$. $H_r^\gamma := \sigma_r^\gamma / {}_nC_r$ is called the r-th anisotropic mean curvature of X. Theorem 6.1 is a generalization of the uniqueness of closed stable CAMC hypersurfaces proved in [2] (CMC case), [14] (γ is of C^∞ and strictly convex), and [15] ($n = 2$, γ is of C^3, and under some assumptions on the Wulff shape and the considered surfaces).

It is interesting to study the uniqueness of closed stable CAMC hypersurfaces for non-convex γ. In §4, we observed that the closed piecewise-C^∞ surfaces shown in Figures 3c and 3d were CAMC for γ_2 and they were convex surfaces. It seems meaningful to check whether they are stable or not.

References

[1] A. D. Alexandrov. A characteristic property of spheres. *Ann. Mat. Pura Appl.* **58** (1962), 303–315.

[2] J. L. Barbosa and M. do Carmo. Stability of hypersurfaces with constant mean curvature. *Math. Z.* **185** (1984), 339–353.

[3] J. L. Barbosa, M. do Carmo and J. Eschenburg. Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. *Math. Z.* **197** (1988), 123–138.

[4] S. Brendle. Embedded self-similar shrinkers of genus 0. *Ann. of Math.* **183** (2016), 715–728.

[5] Y. He, H. Li, H. Ma and J. Ge. Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. *Indiana Univ. Math. J.* **58** (2009), 853–868.

[6] Y. Jikumaru and M. Koiso. Non-uniqueness of closed embedded non-smooth hypersurfaces with constant anisotropic mean curvature. preprint. arXiv:1903.03958 [math.DG]

[7] N. Kapouleas. Complete constant mean curvature surfaces in euclidean three-space. *Ann. of Math.*(2) **131** (1990), 239–330.

[8] N. Kapouleas. Constant mean curvature surfaces constructed by fusing Wente tori. *Invent. Math.* **119** (1995), 443–518.

[9] M. Koiso, Geometry of anisotropic surface energy. *RIMS Kokyuroku* **2068** "Trends in Submanifold Theory"(Ed. by Yu Kawakami), (2018), 57–68.

[10] M. Koiso. Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic mean curvature. preprint. arXiv:1903.03951 [math.DG]

[11] M. Koiso and B. Palmer. Geometry and stability of surfaces with constant anisotropic mean curvature. *Indiana Univ. Math. J.* **54** (2005), 1817–1852.

[12] F. Morgan. Planar Wulff shape is unique equilibrium. *Proc. Amer. Math. Soc.* **133** (2005), 809–813.

[13] P. B. Mucha and P. Rybka. A new look at equilibria in Stefan-type problems in the Plane. *SIAM J. Math. Anal.* **39** (2007), 1120–1134.

[14] B. Palmer. Stability of the Wulff shape. *Proc. Amer. Math. Soc.* **126** (1998), 3661–3667.

[15] B. Palmer. Stable closed equilibria for anisotropic surface energies: surfaces with edges. *J. Geom. Mech.* **4** (2012), 89–97.

[16] R. C. Reilly. The relative differential geometry of nonparametric hypersurfaces. *Duke Math. J.* **43** (1976), 705–721.

[17] J. E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.* **84** (1978), 568–588.

[18] H. C. Wente. Counterexample to a conjecture of H. Hopf. *Pacific J. Math.* **121** (1986), 193–243.

[19] W. L. Winterbottom. Equilibrium shape of a small particle in contact with a foreign substrate. *Acta Metallurgica* **15** (1967), 303–310.

[20] G. Wulff. Zur Frage der Geschwindigkeit des Wachsthums und der Auflösung der Krystallflächen. *Zeitschrift für Krystallographie und Mineralogie* **34** (1901), 449–530.

Institute of Mathematics for Industry
Kyushu University
744 Motooka, Nishi-ku
Fukuoka 819-0395, Japan
E-mail address: koiso@math.kyushu-u.ac.jp

九州大学　マス・フォア・インダストリ研究所　小磯 深幸